INERTIAL RANGE DYNAMICS IN BOUSSINESQ TURBULENCE

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ABSTRACT

L'vov and Falkovich (Physica D **57**) have shown that the dimensionally possible inertial range scaling laws for Boussinesq turbulence, Kolmogorov and Bolgiano scaling, describe steady states with, respectively, constant flux of kinetic energy and of entropy. Following Woodruff (Phys. Fluids **6**), these scaling laws are treated as similarity solutions of the direct interaction approximation for Boussinesq turbulence. The Kolmogorov scaling solution corresponds to a weak perturbation by gravity of a state in which the temperature is a passive scalar but in which a source of temperature fluctuations exists. Using standard inertial range balances, the effective viscosity and conductivity, turbulent Prandtl number, and spectral scaling law constants are computed for Bolgiano scaling.

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I. Introduction

The spectra $E^u(k), E^h(k), E^\theta(k)$ of buoyant turbulence are defined so that

$$\begin{aligned}\frac{1}{2} < u_i u_i > &= \int_0^\infty E^u(k) dk \\ < \theta^2 > &= \int_0^\infty E^\theta(k) dk \\ < u_3 \theta > &= \int_0^\infty E^h(k) dk\end{aligned}$$

where gravity acts in the 3 direction. The variables u_i and θ denote fluctuations about any mean velocity and temperature field. Dimensional analysis suggests two types of inertial range scaling for these spectra. If the gravitational coupling will be neglected, the temperature is a passive scalar and the velocity has the Kolmogorov spectrum

$$E \sim \varepsilon^{2/3} k^{-5/3} \tag{1}$$

where ε is the kinetic energy dissipation rate. Kolmogorov scaling can be expected to apply to wavenumbers k at which nonlinearity dominates the thermal forcing, so that $Re(k) >> Ra(k)$ for appropriately defined scale dependent Reynolds and Rayleigh numbers. Although the limiting temperature distribution is uniform and the spectrum E^θ consequently vanishes in the absence of a source of temperature fluctuations, buoyant flows generally have such a source; in this case, there is a nonzero constant temperature variance dissipation rate N and dimensional analysis gives

$$E^\theta \sim N \varepsilon^{-1/3} k^{-5/3} \tag{2}$$

In Kolmogorov scaling, the inertial range is isotropic, consequently the heat transfer spectrum E^h vanishes.

Bolgiano identified a second possibility[1] in which velocity and temperature fluctuations are determined by N and g, where g denotes the product of thermal expansion coefficient and acceleration of gravity. Then dimensional analysis leads to

$$\begin{aligned}E^u &\sim g^{4/5} N^{2/5} k^{-11/5} \\ E^h &\sim g^{1/5} N^{3/5} k^{-9/5} \\ E^\theta &\sim g^{-2/5} N^{4/5} k^{-7/5}\end{aligned} \tag{3}$$

These scaling laws remained largely a dimensional possibility until L'vov and Falkovich[2] clarified their possible dynamic significance: whereas Kolmogorov scaling corresponds to a steady state inertial range with constant energy flux ε, Bolgiano scaling corresponds to a steady state inertial range with constant entropy flux N; the identification of temperature variance dissipation with entropy flux is due to L'vov.[3] This scaling applies when thermal forcing dominates the nonlinearity so that $Ra(k) >> Re(k)$. A nonzero heat transfer spectrum is possible for Bolgiano scaling, in which the force of gravity introduces a preferred direction. This physical picture might suggest that Kolmogorov scaling will be observed in forced convection and Bolgiano scaling will be observed in free convection; however, arguments[3] that measurements in very high Rayleigh number Rayleigh-Benard convection experiments are consistent with Bolgiano scaling cannot be considered conclusive.[4] Nevertheless, turbulence sustained even at small scales by buoyant forcing is an interesting theoretical possibility which deserves investigation.

This paper considers Kolmogorov and Bolgiano scaling following Woodruff[5] as similarity solutions of the direct interaction approximation (DIA) for buoyant turbulence.[6] It is shown that Kolmogorov scaling can be treated as a perturbation of the passive scalar state in which the gravitational coupling g vanishes. For Bolgiano scaling, this viewpoint results in close connections with the ϵ-expansion of Yakhot and Orszag,[7] which Woodruff interprets as arising from an asymptotic evaluation of the integrated DIA response equation. In Eulerian theories, the ϵ-expansion also serves as a scheme of infrared regularization, which is required since any attempt to compute amplitudes in the spectral scaling laws using the DIA equations directly is defeated by the well-known infrared divergence.[8] The only fully satisfactory solution of this divergence is the formulation of a Lagrangian theory.[9,10] However, the construction of analytical solutions for the Eulerian theory requires some substantial approximations; the added complexity of a Lagrangian theory of any problem with coupled fluctuating fields may justify more or less *ad hoc* modifications of the Eulerian theory like the ϵ-expansion.

II. Direct interaction approximation for Boussinesq turbulence

It will be convenient to write the Boussinesq equations in matrix form as

$$\mathbf{G}_0^{-1}\mathbf{U}(\hat{k}) = \mathbf{\Gamma}(\hat{k}) \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}\ d\hat{q}\ \mathbf{U}(\hat{p})\mathbf{U}(\hat{q}) \tag{4}$$

In Eq. (4), $\mathbf{U}$ is the vector

$$\mathbf{U} = \begin{bmatrix} \mathbf{u} \\ T \end{bmatrix} \tag{5}$$

Written explicitly, the nonlinear term on the right side of Eq. (4) is

$$\Gamma_{imn}(\mathbf{k}) \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}\ d\hat{q}\ U_m(\hat{p}) U_n(\hat{q}) \tag{6}$$

where[6]

$$\Gamma_{imn} = \frac{i}{2} \begin{cases} P_{imn}(\mathbf{k}) & \text{if } i, m, n \neq 4 \\ k_m & \text{if } i = n = 4 \text{ or } i = m = 4 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In Eqs. (4)-(7),

$$\hat{k} = (\Omega, \mathbf{k}) \quad \hat{p} = (\omega, \mathbf{p}) \quad \hat{q} = (\Omega - \omega, \mathbf{q})$$

and

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$
$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$$

The matrix $\mathbf{G}_0$ in Eq. (4) is defined by

$$\mathbf{G}_0(\hat{k})^{-1} = \begin{bmatrix} (-i\Omega + \nu_0 k^2)\mathbf{I} & g\mathbf{P}_3 \\ 0 & (-i\Omega + \kappa_0 k^2) \end{bmatrix} \tag{8}$$

where the vector $\mathbf{P}_3$ has the components P_{i3} and ν_0 and κ_0 are the molecular viscosity and conductivity respectively. Inverting the matrix in Eq. (8) gives the bare Green's matrix

$$\mathbf{G}_0(\hat{k}) = \begin{bmatrix} G_0^u(\hat{k})\mathbf{I} & -G_0^u(\hat{k}) G_0^\theta(\hat{k}) g \mathbf{P}_3 \\ 0 & G_0^\theta(\hat{k}) \end{bmatrix} \tag{9}$$

where G_0^u and G_0^θ are the bare propagators

$$G_0^u(\hat{k}) = (-i\Omega + \nu_0 k^2)^{-1}$$
$$G_0^\theta(\hat{k}) = (-i\Omega + \kappa_0 k^2)^{-1} \tag{10}$$

It should be noted that, unlike the passive scalar equation in which $g = 0$, the Boussinesq equations are effectively nonlinear in T.

The direct interaction approximation (DIA) for Boussinesq turbulence has been derived by Kraichnan.[6] We will attempt to construct an approximate analytical solution to the DIA equations in the inertial range. The descriptors of buoyant turbulence in the DIA are the correlation functions Q^u_{ij}, Q^h_i, Q^T defined by

$$\begin{aligned} < u_i(\hat{k})u_j(\hat{k}') > &= Q^u_{ij}(\hat{k})\delta(\hat{k}+\hat{k}') \\ < u_i(\hat{k})T(\hat{k}') > &= Q^h_i(\hat{k})\delta(\hat{k}+\hat{k}') \\ < T(\hat{k})T(\hat{k}') > &= Q^T(\hat{k})\delta(\hat{k}+\hat{k}') \end{aligned} \tag{11}$$

and the response or Green's functions,

$$\begin{aligned} G^u_{ij}(\hat{k}) &=< \delta u_i(\hat{k})/\delta f^u_j(\hat{k}) > \\ G^{uT}_i(\hat{k}) &=< \delta u_i(\hat{k})/\delta f^T(\hat{k}) > \\ G^{Tu}_j(\hat{k}) &=< \delta T(\hat{k})/\delta f^u_j(\hat{k}) > \\ G^\theta(\hat{k}) &=< \delta T(\hat{k})/\delta f^T(\hat{k}) > \end{aligned}$$

where δf^u_i and δf^T denote small perturbations added to the velocity and temperature equations respectively.

The analysis will be based on the Langevin equation representation of DIA which for an inviscid steady state takes the form

$$\begin{aligned} -i\Omega u_i(\hat{k}) + gT(\hat{k})P_{i3}(\mathbf{k}) &= -\eta^u_{ir}(\hat{k})u_r(\hat{k}) - \eta^{uT}_i(\hat{k})T(\hat{k}) + f^u_i(\hat{k}) \\ -i\Omega T(\hat{k}) &= -\eta^{Tu}_r(\hat{k})u_r(\hat{k}) - \eta^T(\hat{k})T(\hat{k}) + f^T(\hat{k}) \end{aligned} \tag{12}$$

The damping factors η are defined by

$$\begin{aligned} \eta^u_{ir}(\hat{k}) = \frac{1}{4}P_{imn}(\mathbf{k})\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\int_{-\infty}^{\infty} d\omega\ \{&P_{mrs}(\mathbf{p})G^u(\hat{p})Q^u_{ns}(\hat{q}) + P_{nrs}(\mathbf{q})G^u(\hat{q})Q^u_{ms}(\hat{p}) \\ &- p_rG^{uT}_m(\hat{p})Q^h_n(\hat{q}) - q_rG^{uT}_n(\hat{p})Q^h_m(\hat{p})\} \end{aligned} \tag{13}$$

$$\begin{aligned} \eta^{uT}_i(\hat{k}) = -\frac{1}{4}P_{imn}(\mathbf{k})\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\int_{-\infty}^{\infty} d\omega\ \{&p_rG^{uT}_m(\hat{p})Q^u_{rn}(\hat{q}) \\ &+ q_rG^{uT}_n(\hat{q})Q^u_{rm}(\hat{p})\} \end{aligned} \tag{14}$$

$$\begin{aligned} \eta^{Tu}_r(\hat{k}) = \frac{1}{4}\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\int_{-\infty}^{\infty} d\omega\ \ \{&k_np_rG^\theta(\hat{p})Q^h_n(\hat{q}) + k_nq_rG^\theta(\hat{q})Q^h_n(\hat{p}) \\ &+ k_mP_{mrs}(\mathbf{p})G^u(\hat{p})Q^h_s(\hat{q}) + k_mP_{mrs}(\mathbf{q})G^u(\hat{q})Q^h_s(\hat{p}) \\ &- k_nG^{uT}_n(\hat{p})Q^T(\hat{q}) - k_nG^{uT}_n(\hat{q})Q^T(\hat{p}) \\ &- k_mP_{nrs}(\mathbf{q})G^{Tu}_n(\hat{p})Q^u_{ms}(\hat{p}) - k_mP_{nrs}(\mathbf{p})G^{Tu}_n(\hat{q})Q^u_{ms}(\hat{q}) \\ &- k_mq_rG^{Tu}_s(\hat{p})Q^u_{ms}(\hat{p}) - k_mp_rG^{Tu}_s(\hat{q})Q^u_{ms}(\hat{q})\} \end{aligned} \tag{15}$$

$$\eta^T(\hat{k}) = \frac{1}{4}\int_{\mathbf{k=p+q}} d\mathbf{p}\ d\mathbf{q}\int_{-\infty}^{\infty} d\omega\ \{k_n p_m G^\theta(\hat{p})Q^u_{mn}(\hat{q}) + k_n q_m G^\theta(\hat{q})Q^u_{mn}(\hat{p})$$
$$- k_n p_r G^{uT}_n(\hat{p})Q^h_r(\hat{q}) - k_n q_r G^{uT}_n(\hat{q})Q^h_r(\hat{p})\} \tag{16}$$

In Eq. (12), f^u_i and f^T are random forces with the correlations

$$\begin{aligned}
< f^u_i(\hat{k})f^u_j(-\hat{k}) > &= F^{uu}_{ij}(\hat{k}) \\
< f^u_i(\hat{k})T(-\hat{k}) > &= F^{uT}_i(\hat{k}) \\
< T(\hat{k})T(-\hat{k}) > &= F^{TT}(\hat{k})
\end{aligned}$$

where

$$\begin{aligned}
F^{TT}(\hat{k}) &= k_m k_r \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}\ d\hat{q}\ Q^u_{mr}(\hat{p})Q^T(\hat{q}) + Q^h_m(\hat{p})Q^h_r(\hat{q}) \\
F^{uT}_i(\hat{k}) &= k_n P_{irs}(\mathbf{k})\int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}\ d\hat{q}\ Q^h_r(\hat{p})Q^u_{ns}(\hat{q}) \qquad (17)\\
F^{uu}_{ij}(\hat{k}) &= P_{imn}(\mathbf{k})P_{jrs}(\mathbf{k})\int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}\ d\hat{q}\ Q^u_{mr}(\hat{p})Q^u_{ns}(\hat{q})
\end{aligned}$$

The DIA theory is completed by the definitions Eq. (11) of the correlations in terms of the fields u, T and by the equations of motion for the G's, which are given by

$$\begin{aligned}
&-i\Omega G^u_{ij}(\hat{k}) + gG^{Tu}_j(\hat{k})P_{i3}(\mathbf{k}) + \eta^{uT}_i G^{Tu}_j(\hat{k}) + \eta^u_{ip}(\hat{k})G^u_{pj} = 1 \\
&-i\Omega G^{uT}_i(\hat{k}) + gG^T(\hat{k})P_{i3}(\mathbf{k}) + \eta^{uT}_i(\hat{k})G^T(\hat{k}) + \eta^u_{ip}(\hat{k})G^{uT}_p(\hat{k}) = 0 \\
&-i\Omega G^{Tu}_j(\hat{k}) + \eta^T(\hat{k})G^{Tu}_j(\hat{k}) + \eta^{Tu}_r(\hat{k})G^u_{rj}(\hat{k}) = 0 \\
&-i\Omega G^\theta(\hat{k}) + \eta^T(\hat{k})G^\theta(\hat{k}) + \eta^{Tu}_r(\hat{k})G^{uT}_r(\hat{k}) = 1 \qquad (18)
\end{aligned}$$

These equations are simply the conditions that the solution of the Langevin equations, Eq. (12) is given by

$$\mathbf{U}(\hat{k}) = \mathbf{G}(\hat{k})\begin{bmatrix} \mathbf{f}^u(\hat{k}) \\ f^T(\hat{k}) \end{bmatrix} \tag{19}$$

in the matrix notation of Eqs. (4)-(9). In Eq. (19), $\mathbf{G}$ is the renormalized response matrix

$$\mathbf{G}(\hat{k}) = \begin{bmatrix} G_{ij}(\hat{k}) & G^{uT}_i(\hat{k}) \\ G^{Tu}_j(\hat{k}) & G^\theta(\hat{k}) \end{bmatrix}$$

The notation means that G^{Tu} is a row and G^{uT} is a column.

The DIA equations are considerably more complicated than the bare equations; although the coupling η^{uT} can be considered a renormalized gravitational coupling, η^{Tu} has no analog in the bare equation. Moreover, whereas the bare theory contains only the two response functions of Eq. (10), DIA adds the functions G^{uT} and G^{Tu}. Analytical progress will clearly require rational simplification of these equations. It will be convenient to solve the DIA equations by an iterative procedure, starting with an *ansatz* for the G's and F's. When these are known, the correlations can be computed from Eqs. (11) and (19). Knowing G and Q, substitute in Eqs. (13)-(16) to obtain the damping factors. This procedure can now be iterated, since an updated $\mathbf{G}$ can be found by solving Eq. (18), and updated forces can be computed from Eqs. (17).

The initial assumption for the Green's matrix will be

$$\mathbf{G}(\hat{k}) = \begin{bmatrix} G^u(\hat{k})\mathbf{I} & -G^u(\hat{k})G^\theta(\hat{k})g\mathbf{P}_3 \\ 0 & G^\theta(\hat{k}) \end{bmatrix} \tag{20}$$

so that the bare and renormalized Green's matrices have the same structure, and are determined by two scalar Green's functions $G^u(\hat{k})$ and $G^\theta(\hat{k})$ alone. This type of diagonal approximation is useful in treating coupled field problems. Leaving the random force general temporarily, the correlation matrix will be determined by forming correlations of the amplitudes found from Eqs. (11) and (19). The explicit expressions are

$$\begin{aligned} Q^u_{ij}(\hat{k}) &= G^u(\hat{k})G^u(-\hat{k})F^{uu}_{ij}(\hat{k}) \\ &- gG^u(\hat{k})G^u(-\hat{k})G^\theta(-\hat{k})P_{j3}(\mathbf{k})F^{uT}_i(\hat{k}) \\ &- gG^u(-\hat{k})G^u(\hat{k})G^\theta(\hat{k})P_{i3}(\mathbf{k})F^{uT}_j(\hat{k}) \\ &+ g^2G^u(\hat{k})G^\theta(\hat{k})G^u(-\hat{k})G^\theta(-\hat{k})P_{i3}(\mathbf{k})P_{j3}(\mathbf{k})F^{TT}(\hat{k}) \\ Q^h_i(\hat{k}) &= G^u(\hat{k})G^\theta(-\hat{k})F^{uT}_i(\hat{k}) - gG^u(\hat{k})G^\theta(\hat{k})G^\theta(-\hat{k})P_{i3}(\mathbf{k})F^{TT}(\hat{k}) \\ Q^T(\hat{k}) &= G^\theta(\hat{k})G^\theta(-\hat{k})F^{TT}(\hat{k}) \end{aligned} \tag{21}$$

Now we will adopt for Kolmogorov scaling the *ansatz*

$$F^{uu}_{ij}(\hat{k}) = P_{ij}(\mathbf{k})C^u_F\varepsilon k^{-3} \quad F^{uT} = F^{TT} = 0 \tag{22}$$

and for Bolgiano scaling

$$F^{TT}(\hat{k}) = C^T_F N k^{-3} \quad F^{uu} = F^{uT} = 0 \tag{23}$$

The forces in Eqs. (22) and (23) are white noise in time. It will be shown later that a corrected force can be derived which is not white noise in time. Thus, DIA predicts that the effective forcing is in fact certainly not white noise in time; the introduction of such a forcing is merely used to provide an initial approximation to the solution of the DIA equations in the inertial range.

A. *Kolmogorov scaling*

Substitution of Eq. (22) in Eq. (21) results in

$$\begin{aligned} Q^u_{ij}(\hat{k}) &= G^u(\hat{k})G^u(-\hat{k})P_{ij}(\mathbf{k})C^u_F\varepsilon k^{-3} \\ Q^h_i(\hat{k}) &= 0 \\ Q^T(\hat{k}) &= 0 \end{aligned} \tag{24}$$

These are the correlations appropriate to a purely passive scalar. The vanishing of the Q^T correlation reflects the absence of a source for temperature fluctuations: the long time temperature distribution is uniform and therefore vanishingly small.

Substituting Eq. (24) in Eq. (13) leads to the isotropic renormalized damping

$$\eta^u_{ij}(\hat{k}) = \eta^u(\hat{k})P_{ij}(\mathbf{k})$$

where

$$\eta^u(\hat{k}) = \frac{1}{4}P_{imn}(\mathbf{k})\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} P_{mrs}(\mathbf{p})P_{ns}(\mathbf{q})\int_{-\infty}^{\infty} d\omega\; G^u(\hat{p})G^u(\hat{q})G^u(-\hat{q})C^u_F\varepsilon q^{-3} \tag{25}$$

Substituting Eq. (24) in Eq. (14) leads to the renormalized diffusivity

$$\eta^T(\hat{k}) = \frac{1}{4}k_nk_m\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} P_{nm}(\mathbf{q})\int_{-\infty}^{\infty} d\omega\; G^\theta(\hat{p})G^u(\hat{q})G^u(-\hat{q})C^u_F\varepsilon q^{-3} \tag{26}$$

Substituting Eq. (24) in Eq. (11) leads to the renormalized gravitational coupling

$$\eta^{uT}_i(\hat{k}) = \eta^{uT}(\hat{k})P_{i3}(\mathbf{k})$$

where

$$\begin{aligned} \eta^{uT}(\hat{k}) = -\frac{1}{4}gP_{3mn}(\mathbf{k})&\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} p_rP_{m3}(\mathbf{p})P_{rn}(\mathbf{q})\times \\ &\int_{-\infty}^{\infty} d\omega\; G^u(\hat{p})G^\theta(\hat{p})G^u(\hat{q})G^u(-\hat{q})C^u_F\varepsilon q^{-3} \end{aligned} \tag{27}$$

The coupling η^{Tu} vanishes identically.

In this approximation which neglects G^{Tu} and G^{uT}, the system of equations Eq. (18) for the response functions is replaced by the simplified system

$$
\begin{aligned}
-i\Omega G^u(\hat{k}) + \eta^u(\hat{k}) G^u(\hat{k}) &= 1 \\
-i\Omega G^\theta(\hat{k}) + \eta^T(\hat{k}) G^\theta(\hat{k}) &= 1
\end{aligned}
\tag{28}
$$

The couplings η^{uT} and η^{Tu} do not appear in these simplified response equations, and can therefore be ignored.

It is shown in the Appendix that the assumption of white noise in time forcing is consistent with Markovian damping in which

$$
\begin{aligned}
\eta^u(\hat{k}) &= \eta^u(k) \\
\eta^T(\hat{k}) &= \eta^T(k)
\end{aligned}
$$

The corresponding response functions are exponential:

$$
\begin{aligned}
G^u(\hat{k}) &= (-i\Omega + \eta(k))^{-1} \\
G^\theta(\hat{k}) &= (-i\Omega + a\eta(k))^{-1}
\end{aligned}
\tag{29}
$$

The damping factors $\eta(k)$ and $\eta^T(k)$ for Kolmogorov scaling are

$$
\begin{aligned}
\eta(k) &= C_D \varepsilon^{1/3} k^{2/3} \\
\eta^T(k) &= a C_D \varepsilon^{1/3} k^{2/3}
\end{aligned}
\tag{30}
$$

The constant C_D is to be calculated from the theory. Eq. (30) assumes the existence of a scale independent turbulent Prandtl number $1/a$, which is also to be computed from the theory. Substitution of Eqs. (29) and (30) in Eqs. (25) and (16), evaluating the integrals by the ϵ-expansion following Woodruff[6] leads to the results for the passive scalar previously obtained by Yakhot and Orszag.[7] In view of the vanishing of Q^T and Q^h, Eq. (17) shows that the assumption Eq. (22) for the random force was consistent with DIA.

The passive scalar solution obtained in Eqs. (25)-(26) can be corrected by adding a thermal force to Eq. (22):

$$
F^{uu}_{ij}(\hat{k}) = P_{ij}(\mathbf{k}) C^u_F \varepsilon k^{-3} \quad F^{uT} = 0 \quad F^{TT} = C^T_F N k^{-3}
$$

and treating g as an expansion parameter. Substituting this forcing in Eq. (21) and integrating over ω leads to $O(g^2)$ corrections to the correlation Q^u and to the results

$$Q^h \sim g\frac{N}{\varepsilon^{2/3}}k^{-7/3}$$
$$Q^T \sim \frac{N}{\varepsilon^{1/3}}k^{-5/3}$$

The result for Q^T is equivalent to Eq. (2). In this analysis, nonvanishing coupling to the velocity equation is needed to produce a nonvanishing temperature spectrum.

B. *Bolgiano scaling*

Substituting Eq. (17) in Eq. (15) leads for Bolgiano scaling to the correlations

$$\begin{aligned}
Q^u_{ij}(\hat{k}) &= g^2 G^u(\hat{k})G^u(-\hat{k})G^\theta(\hat{k})G^\theta(-\hat{k})P_{i3}(\mathbf{k})P_{j3}(\mathbf{k})C_F^T N k^{-3} \\
Q^h_i(\hat{k}) &= -gG^u(\hat{k})G^\theta(\hat{k})G^\theta(-\hat{k})P_{i3}(\mathbf{k})C_F^T N k^{-3} \\
Q^T(\hat{k}) &= G^\theta(\hat{k})G^\theta(-\hat{k})C_F^T N k^{-3}
\end{aligned} \tag{31}$$

Eqs. (31) imply various relations among inertial range spectra which can be derived by integrating over frequency to obtain single time correlations. Again assuming Markovian forms Eq. (29) for the response functions G and G^θ, the single time correlations are found to be

$$\begin{aligned}
Q^T(\mathbf{k}) &= \int_{-\infty}^{\infty} d\omega\ Q^T(\hat{k}) = \frac{C_F^T N k^{-3}}{2a\eta(k)} \\
Q^h_i(\mathbf{k}) &= \int_{-\infty}^{\infty} d\omega\ Q^h(\hat{k}) = -gP_{i3}(\mathbf{k})\frac{C_F^T N k^{-3}}{2a(a+1)\eta(k)^2} \\
&= -\frac{g}{(a+1)\eta(k)}P_{i3}(\mathbf{k})Q^T(\mathbf{k}) \\
Q^u_{ij}(\mathbf{k}) &= \int_{-\infty}^{\infty} d\omega\ Q^T(\hat{k}) = \frac{g^2}{2a(a+1)}\frac{C_F^T N k^{-3}}{\eta(k)^3}P_{i3}(\mathbf{k})P_{j3}(\mathbf{k}) \\
&= \frac{g^2}{(a+1)\eta(k)^2}P_{i3}(\mathbf{k})P_{j3}(\mathbf{k})Q^T(\mathbf{k})
\end{aligned} \tag{32}$$

Thus, all of the single time correlations can be expressed in terms of $Q^T(\mathbf{k})$. Substituting these correlation and response functions in Eqs. (13)-(16), we obtain the renormalized quantities

$$\eta(\hat{k}) = \frac{1}{4}g^2 P_{imn}(\mathbf{k})\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}}\{p_r P_{m3}(\mathbf{p})P_{n3}(\mathbf{q})C_F^T N q^{-3}\times$$

$$\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^\theta(\hat p)G^u(\hat q)G^\theta(\hat q)G^\theta(-\hat q)$$

$$+ P_{mrs}(\mathbf{p})P_{n3}(\mathbf{q})P_{s3}(\mathbf{q})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^u(\hat q)G^u(-\hat q)G^\theta(\hat q)G^\theta(-\hat q)\Big\} \quad (33)$$

$$\eta_i^{uT}(\hat k) = -\frac{1}{4} g P_{imn}(\mathbf{k})\int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} p_r P_{m3}(\mathbf{p})P_{r3}(\mathbf{q})P_{n3}(\mathbf{q})C_F^T N q^{-3}\times$$
$$\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^\theta(\hat p)G^u(\hat q)G^u(-\hat q)G^\theta(\hat q)G^\theta(-\hat q) \quad (34)$$

$$\eta^T(\hat k) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} g^2 k_n p_m P_{m3}(\mathbf{q})P_{n3}(\mathbf{q})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^\theta(\hat p)G^u(\hat q)G^u(-\hat q)G^\theta(\hat q)G^\theta(-\hat q)$$
$$+ g^2 k_n p_r P_{n3}(\mathbf{p})P_{r3}(\mathbf{q})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^\theta(\hat p)G^u(\hat q)G^\theta(\hat q)G^\theta(-\hat q) \quad (35)$$

$$\eta_m^{Tu}(\hat k) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} g k_n p_m P_{n3}(\mathbf{q})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^\theta(\hat p)G^u(\hat q)G^\theta(\hat q)G^\theta(-\hat q)$$
$$+ g k_r P_{rms}(\mathbf{p})P_{s3}(\mathbf{q})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^u(\hat q)G^\theta(\hat q)G^\theta(-\hat q)$$
$$+ g k_n p_m P_{n3}(\mathbf{p})C_F^T N q^{-3}\int_{-\infty}^{\infty} d\omega\ G^u(\hat p)G^\theta(\hat p)G^\theta(\hat p)G^\theta(\hat q)G^\theta(-\hat q) \quad (36)$$

As in the case of Kolmogorov scaling, the couplings η^{Tu} and η^{uT} can be ignored at this level of approximation.

III. Determination of inertial range constants for Bolgiano scaling

A. *The turbulent Prandtl number*

The procedure for determining inertial range constants is standard.[11] Because of Eq. (32), there are only three inertial range constants, namely the constant C_D in the time scale defined for Bolgiano scaling by

$$\eta(k) = C_D g^{2/5} N^{1/5} k^{2/5} \quad (37)$$

the turbulent Prandtl number $1/a$ defined so that the thermal damping function $\eta^T(k)$ is

$$\eta^T(k) = aC_D g^{2/5} N^{1/5} k^{2/5} \quad (38)$$

and the constant C_B in the spectral scaling law

$$E^T(k) = C_B g^{-2/5} N^{4/5} k^{-7/5} \tag{39}$$

Setting $\Omega = 0$ in the approximate response equations Eq. (28),

$$\begin{aligned} G^u(\hat{k})\eta^u(\hat{k}) \mid_{\Omega=0} &= 1 \\ G^\theta(\hat{k})\eta^T(\hat{k}) \mid_{\Omega=0} &= 1 \end{aligned} \tag{40}$$

Performing the frequency integrations in Eq. (40),

$$\begin{aligned} 1 = &\frac{g^2}{4\eta(k)} P_{rmn}(\mathbf{k}) \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p} d\mathbf{q} \times \\ &\frac{a\eta(p) + (2a^2 + 2a + 1)\eta(q)}{a(a+1)\eta(q)(\eta(p)+\eta(q))(\eta(q)+a\eta(p))(\eta(p)+a\eta(q))} p_r P_{m3}(\mathbf{p}) P_{n3}(\mathbf{q}) Q^T(q) \\ &+ \frac{(a+1)\eta(q) + \eta(p)}{(a+1)\eta(q)^2(\eta(p)+\eta(q))(a\eta(q)+\eta(p))} P_{mrs}(\mathbf{p}) P_{n3}(\mathbf{q}) P_{s3}(\mathbf{q}) Q^T(q) \end{aligned} \tag{41}$$

$$\begin{aligned} 1 = &\frac{g^2}{4a\eta(k)} \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p} d\mathbf{q} \times \\ &\frac{a\eta(p) + (a+1)\eta(q)}{a(a+1)\eta(q)^2(a\eta(p)+\eta(q))(\eta(p)+\eta(q))} k_n p_m P_{m3}(\mathbf{q}) P_{n3}(\mathbf{q}) Q^T(q) \\ &+ [a(a+1)\eta(q)(\eta(p)+\eta(q))(a\eta(p)+\eta(q))(\eta(p)+a\eta(q))]^{-1} \times \\ &k_n p_r P_{n3}(\mathbf{p}) P_{r3}(\mathbf{q}) Q^T(q) \end{aligned} \tag{42}$$

Substitution in Eqs. (41) and (42) of the inertial range forms Eqs. (3) leads to the usual problem that the integrals are infrared divergent. We follow Woodruff's treatment of this problem for hydrodynamic turbulence.[5] We begin by seeking scaling forms for E^θ and η such that the right hand sides of Eqs. (41)-(42) are ultraviolet divergent. The scalings are found by assuming that

$$\eta \sim k^{2-\epsilon/5}$$

The right sides of Eqs. (41) and (42) must be scale independent. Thus, if $Q(k) \sim k^\Delta$, necessarily $\Delta = 3 - 4\epsilon/5$ so that

$$\begin{aligned} \eta(k) &\sim k^{2-\epsilon/5} \\ E^\theta(k) &\sim k^{5-4\epsilon/5} \end{aligned} \tag{43}$$

Bolgiano scaling, defined by Eqs. (3), (37), and (38) corresponds to $\epsilon = 8$. Substituting the scaling forms of Eq. (43) in Eqs. (41) and (42) shows that the integrals are ultraviolet divergent for $\epsilon \leq 0$ and logarithmic when $\epsilon = 0$. The values of the integrals when $\epsilon > 0$ can be approximated by asymptotic expansion about $\epsilon = 0$. Because of the ultraviolet divergence, the integrals are dominated in this case by distant interactions in which $k/p, k/q \to 0$ and $p, q \to \infty$. Substituting the scaling forms Eqs. (43) in Eqs. (41) and (42), evaluating the integrals in the distant interaction limit, and performing the spherical integrations leads after lengthy but straightforward calculations to the results

$$\begin{aligned}&\frac{g^2}{4\eta(k)} P_{rmn}(\mathbf{k}) \int^* \frac{a\eta(p) + (2a^2 + 2a + 1)\eta(q)}{a(a+1)\eta(q)(\eta(p)+\eta(q))(\eta(q)+a\eta(p))(\eta(p)+a\eta(q))} \times \\ &\quad p_r P_{m3}(\mathbf{p}) P_{n3}(\mathbf{q}) Q^T(q) \\ &\quad = \frac{1}{16}\frac{C_B}{C_D^4}[a^2(a+1)^3]^{-1}\frac{2}{15}(6a^2+5a+3)\frac{5}{6\epsilon}\end{aligned} \tag{44}$$

$$\begin{aligned}&\frac{g^2}{4\eta(k)} P_{rmn}(\mathbf{k}) \int^* \frac{(a+1)\eta(q) + \eta(p)}{(a+1)\eta(q)^2(\eta(p)+\eta(q))(a\eta(q)+\eta(p))} \times \\ &\quad P_{mrs}(\mathbf{p}) P_{n3}(\mathbf{q}) P_{s3}(\mathbf{q}) Q^T(q) \\ &\quad = \frac{1}{16}\frac{C_B}{C_D^4}[a(a+1)^3]^{-1}\frac{2}{15}(6a^2+18a+16)\frac{5}{6\epsilon}\end{aligned} \tag{45}$$

$$\begin{aligned}&\frac{g^2}{4a\eta(k)} P_{rmn}(\mathbf{k}) \int^* \frac{a\eta(p) + (a+1)\eta(q)}{a(a+1)\eta(q)^2(a\eta(p)+\eta(q))(\eta(p)+\eta(q))} \times \\ &\quad k_n p_m P_{m3}(\mathbf{q}) P_{n3}(\mathbf{q}) Q^T(q) \\ &\quad = \frac{1}{16}\frac{C_B}{C_D^4}[a^3(a+1)^2]^{-1}\frac{2}{3}(2a+1)\frac{5}{6\epsilon}\end{aligned} \tag{46}$$

$$\begin{aligned}&\frac{g^2}{4\eta(k)} P_{rmn}(\mathbf{k}) \int^* \frac{a\eta(p) + (2a^2 + 2a + 1)\eta(q)}{a(a+1)\eta(q)(\eta(p)+\eta(q))(a\eta(p)+\eta(q))(\eta(p)+a\eta(q))} \times \\ &\quad k_n p_r P_{n3}(\mathbf{p}) P_{r3}(\mathbf{q}) Q^T(q) \\ &\quad = \frac{1}{16}\frac{C_B}{C_D^4}[a^3(a+1)^2]^{-1}\frac{2}{3}\frac{5}{6\epsilon}\end{aligned} \tag{47}$$

where

$$\int^* = \frac{1}{2}\left[\int_{\mathbf{q}=\mathbf{k}-\mathbf{p}, p \geq k} d\mathbf{p} + \int_{\mathbf{p}=\mathbf{k}-\mathbf{q}, q \geq k} d\mathbf{q}\right]$$

Note that the symmetrization in $\int^*$ is required because the original integration over wavevector triads is certainly symmetric in $\mathbf{p}$ and $\mathbf{q}$, but $\int_{\mathbf{q}=\mathbf{k}-\mathbf{p},p\geq k} d\mathbf{p}$ by itself for example is not symmetric. The restrictions $p \geq k$ and $q \geq k$ in $\int^*$ are infrared regularizations which follow from the ϵ-expansion.[5] This regularization suppresses a crossover to a dominant sweeping interaction when $\epsilon = 7$ which reflects the infrared divergence characteristic of Eulerian theories.[6] Evaluation of the integrals can be simplified by summing over the direction of gravity, which always enters these integrals in pairs.

Substituting the results Eqs. (44)-(47) in the integrated response equations Eqs. (41) and (42) leads finally to

$$\begin{aligned} \frac{C_D^4}{C_B} &= \frac{1}{16}[a^2(a+1)^3]^{-1}\frac{5}{6\epsilon}\frac{1}{15}(12a^3+48a^2+42a+6) \\ \frac{C_D^4}{C_B} &= \frac{1}{16}[a^3(a+1)]^{-1}\frac{5}{6\epsilon}\frac{2}{3}4 \end{aligned} \tag{48}$$

Equating the right hand sides of Eqs. (48) leads to an equation in a alone. Clearing fractions in this equation,

$$12a^4 + 48a^3 + 2a^2 - 74a - 40 = 0$$

which has a unique positive root

$$a \sim 1.275$$

Now substituting in Eq. (48), we find

$$\frac{C_D^4}{C_B} \equiv \mathcal{A} = 0.0037 \tag{49}$$

B. *Evaluation of the constants C_D and C_B*

Following Kraichnan,[12] a second relation between C_D and C_B is found from the constancy of inertial range entropy flux defined[3] by

$$N = \int_{k\leq k_0} d\mathbf{k} \int_{p,q\geq k_0} d\mathbf{p}\ d\mathbf{q}\ S(\mathbf{k},\mathbf{p},\mathbf{q}) - \int_{2k_0\geq k\geq k_0} d\mathbf{k} \int_{p,q\leq k_0} d\mathbf{p}\ d\mathbf{q}\ S(\mathbf{k},\mathbf{p},\mathbf{q}) \tag{50}$$

where the entropy flux density S is

$$S(\mathbf{k},\mathbf{p},\mathbf{q}) = -ik_m < u_m(\mathbf{p},t)T(\mathbf{q},t)T(-\mathbf{k},t) >$$

The flux N should be finite and independent of k_0 in the inertial range. In the DIA closure,

$$S(\mathbf{k},\mathbf{p},\mathbf{q}) = \lim_{t\to\infty}\Sigma(\mathbf{k},\mathbf{p},\mathbf{q},t)$$

where, reverting to time domain notation for simplicity,

$$\begin{aligned}
\Sigma(\mathbf{k},\mathbf{p},\mathbf{q},t) = -\frac{1}{2}k_m \int_0^t \; ds \; \big\{ & G^u(\mathbf{p},t,s)P_{mrs}(\mathbf{p})Q^h_r(\mathbf{q},s,t)Q^h_s(\mathbf{k},s,t) \\
& + G^\theta(\mathbf{q},t,s)q_r\big[Q^h_m(\mathbf{p},t,s)Q^h_r(\mathbf{k},s,t) + Q^u_{mr}(\mathbf{p},t,s)Q^T(\mathbf{k},t,s)\big] \\
& - G^\theta(\mathbf{k},t,s)k_r\big[Q^h_m(\mathbf{p},t,s)Q^h_r(\mathbf{q},s,t)\big] + Q^u_{mr}(\mathbf{p},t,s)Q^T(\mathbf{q},t,s)\big]\big\} \\
& + \frac{1}{2}g k_m \int_0^t \; ds \; \int_0^s \; dr \times \\
& \big\{p_r G^\theta(p,s,r)\big[Q^h_r(\mathbf{q},r,t)Q^T(\mathbf{k},r,t) + Q^h(\mathbf{k},r,t)Q^T(\mathbf{q},r,t)\big]\big\} \\
& + \frac{1}{2}g k_m \int_0^t \; ds \; \int_0^t \; dr \times \\
& + \big\{q_r G^\theta(q,t,r)\big[Q^h_r(\mathbf{p},r,s)Q^T(\mathbf{k},r,t) + Q^h_r(\mathbf{k},r,t)Q^T(\mathbf{p},r,s)\big] \\
& - k_r G^\theta(k,t,s)\big[Q^h_r(\mathbf{p},r,s)Q^T(\mathbf{q},r,t) + Q^h_r(\mathbf{q},r,t)Q^T(\mathbf{p},r,s)\big]\big\}
\end{aligned}$$

The time integrations are performed using fluctuation dissipation relations, which follow from Eqs. (29) and (19)

$$< u_i(\mathbf{k},t_1)T(\mathbf{k}',t_2) >= \delta(\mathbf{k}+\mathbf{k}')\begin{cases} G^u(\mathbf{k},t_1,t_2)Q^h(k) & \text{for } t_1 > t_2 \\ G^\theta(\mathbf{k},t_2,t_1)Q^h(k) & \text{for } t_2 > t_1 \end{cases}$$

Then a lengthy but straightforward calculation leads to

$$\begin{aligned}
S(\mathbf{k},\mathbf{p},\mathbf{q}) = & -\frac{1}{2}\Theta_1(k,p,q)\big\{ \; k_m P_{mrs}(\mathbf{p})P_{r3}(\mathbf{q})P_{s3}(\mathbf{k})Q^h(k)Q^h(q) \\
& + P_{mr}(\mathbf{p})k_m Q^u(p)\big[q_r Q^T(k) - k_r Q^T(q)\big] \\
& + P_{ml}(\mathbf{p})k_m Q^h(p)\big[q_r P_{r3}(\mathbf{k})Q^h(k) - k_r P_{r3}(\mathbf{q})q^h(q)\big]\big\} \\
& + \frac{1}{2}g k_m P_{m3}(\mathbf{p})\big\{\Theta_1(k,p,q)\Theta_2(k,p,q)\big[p_r P_{r3}(\mathbf{q})Q^h(q)Q^T(k) + p_r P_{r3}(\mathbf{k})Q^h(k)Q^T(q)\big] \\
& + \big[\Theta_1(k,p,q)\Theta_3(k,p,q) + (2\eta(p))^{-1}\Theta_1(k,p,q)\big]q_r P_{r3}(\mathbf{p})Q^h(p)Q^T(k) \\
& + \big[\Theta_1(k,p,q)\Theta_2(k,p,q) + ((1+a)\eta(p))^{-1}\Theta_1(k,p,q)\big]q_r P_{r3}(\mathbf{k})Q^h(k)Q^T(p) \\
& - \big[\Theta_1(k,p,q)\Theta_2(k,p,q) + (2\eta(p))^{-1}\Theta_1(k,p,q)\big]k_r P_{r3}(\mathbf{p})Q^h(p)Q^T(q) \\
& - \big[\Theta_1(k,p,q)\Theta_2(k.p,q) + ((1+a)\eta(p))^{-1}\Theta_1(k,p,q)\big]k_r P_{r3}(\mathbf{q})Q^h(q)Q^T(p)\big\}
\end{aligned} \tag{51}$$

where the time scales $\Theta_i(k,p,q)$ are defined by

$$\begin{aligned}
\Theta_1(k,p,q) &= [\eta(p) + a\eta(q) + a\eta(k)]^{-1} \\
\Theta_2(k,p,q) &= a^{-1}[\eta(p) + \eta(q) + \eta(k)]^{-1} \\
\Theta_3(k,p,q) &= [a\eta(p) + \eta(q) + a\eta(k)]^{-1}
\end{aligned} \tag{52}$$

Note that the time scales in the g dependent term in Eq. (51) have a nonstandard form, which might not suggest itself in a phenomenological approach like EDQNM.

Now Eqs. (37)-(39) can be substituted in Eq. (51), with Eq. (32) used to express Q^h and Q^u in terms of Q^T alone. To evaluate the flux integral Eq. (50), define the quantity $T(k,p,q)$ by

$$\int dp\ dq\ T(k,p,q) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ S(\mathbf{k},\mathbf{p},\mathbf{q})$$

The scalings are such that

$$T(\lambda k, \lambda p, \lambda q) = \lambda^{-3} T(k,p,q)$$

a result equivalent to the choice of a k^{-3} Langevin force in Eq. (23). This force scaling is also equivalent to Orszag's condition[13] on the scaling laws for Θ and Q that a stationary inertial range is possible. Accordingly, the -3 forcing appears to be universal, a suggestion made on other grounds in Ref. 14. This scaling also implies that the inertial range transfer does not depend on the wavenumber k_0 and permits the transfer integral to be reduced to a double integral over a finite region following Kraichnan.[12] Numerical integration of Eqs. (21)-(22) leads to

$$\frac{C_B^2}{C_D^3} \equiv \mathcal{B} = 0.21 \tag{53}$$

Eqs. (49) and (53) imply tentative values of the inertial range constants

$$\begin{aligned}
C_B &= \mathcal{A}^{3/5}\mathcal{B}^{4/5} = 0.01 \\
C_D &= \mathcal{A}^{2/5}\mathcal{B}^{1/5} = 0.08
\end{aligned}$$

Whereas the turbulent Prandtl number $1/a = 0.78$ is not far from the passive scalar value, these constants depart markedly from values for Kolmogorov scaling. Further investigation of the inertial range constants is indicated.

IV. Iterative solution of DIA

The approximation considered so far replaced the exact system of equations for the response functions Eq. (18) by the simplified set Eq. (28) by ignoring the response functions G^{Tu} and G^{uT}. This approximation also entails ignoring the couplings functions η^{Tu} and η^{uT}. But Eqs. (13)-(16) provide expressions for η^{Tu} and η^{uT}. As noted previously, these can be substituted in Eqs. (18) to compute corrected response functions including G^{Tu} and G^{uT}. The force correlations can be corrected similarly. It is evident that this iteration will not alter the scalings, but changes to the computed inertial range constants cannot be ruled out in advance.

To indicate the general character of this iterative procedure more explicitly, it will be convenient to simplify the spatial dependence of the structure of Eq. (18). Assume that the solution has the form

$$\begin{aligned}
G^u_{ij}(\hat{k}) &= G^u_1(\hat{k})P_{ij}(\mathbf{k}) + G^u_2(\hat{k})P_{i3}(\mathbf{k})P_{j3}(\mathbf{k}) \\
G^{uT}_i(\hat{k}) &= G^{uT}(\hat{k})P_{i3}(\mathbf{k}) \\
G^{Tu}_j(\hat{k}) &= G^{Tu}(\hat{k})P_{3j}(\mathbf{k})
\end{aligned} \tag{54}$$

The results from Eqs. (33)-(36) have the corresponding forms

$$\begin{aligned}
\eta^u_{ij}(\hat{k}) &= \eta^u_1(k)P_{ij}(\mathbf{k}) + \eta^u_2(k)P_{i3}(\mathbf{k})P_{j3}(\mathbf{k}) \\
\eta^{uT}_i(\hat{k}) &= \eta^{uT}(k)P_{i3}(\mathbf{k}) \\
\eta^{Tu}_j(\hat{k}) &= \eta^{Tu}(k)P_{3j}(\mathbf{k})
\end{aligned} \tag{55}$$

where the Markovian character of the approximation has been used. Now substitute Eqs. (54) and (55) in Eq. (18) and note, as observed previously, that the index 3 corresponding to the direction of gravity, always appears in obvious pairs. The isotropic part of the solution can be evaluated by summing over this index.[15] This procedure leads to a system of linear scalar equations with a simple structure

$$\begin{aligned}
[-i\Omega + \eta^u(k)]G^u(\hat{k}) + \eta^{uT}(k)G^{Tu}(\hat{k}) &= 1 \\
[-i\Omega + \eta^u(k)]G^{uT}(\hat{k}) + \eta^{uT}(k)G^{T}(\hat{k}) &= 0 \\
[-i\Omega + \eta^T(k)]G^{Tu}(\hat{k}) + \eta^{Tu}(k)G^{u}(\hat{k}) &= 0 \\
[-i\Omega + \eta^T(k)]G^{T}(\hat{k}) + \eta^{Tu}(k)G^{uT}(\hat{k}) &= 1
\end{aligned} \tag{56}$$

where $G^u = G_1^u + G_2^u$ and $\eta^u = \eta_1^u + \eta_2^u$. The notation in Eq. (56) has been simplified by including the bare gravitational coupling in η^{uT}. Eq. (56) can be solved for the response functions:

$$
\begin{aligned}
G^u(\hat{k}) &= [-i\Omega + \eta^T]/[(-i\Omega + \eta^u)(-i\Omega + \eta^T) - \eta^{uT}\eta^{Tu}] \\
G^{uT}(\hat{k}) &= -\eta^{uT}/[(-i\Omega + \eta^u)(-i\Omega + \eta^T) - \eta^{uT}\eta^{Tu}] \\
G^{Tu}(\hat{k}) &= -\eta^{Tu}/[(-i\Omega + \eta^u)(-i\Omega + \eta^T) - \eta^{uT}\eta^{Tu}] \\
G^\theta(\hat{k}) &= [-i\Omega + \eta^u]/[(-i\Omega + \eta^u)(-i\Omega + \eta^T) - \eta^{uT}\eta^{Tu}]
\end{aligned} \tag{57}
$$

where the damping functions η are found by substituting the lowest order response functions Eq. (29) in Eqs. (33)-(36) and setting $\Omega = 0$. Whereas the lowest order response functions are rational functions of first order, the corrected response functions of Eq. (57) are rational functions of second order with poles at the points

$$
\Omega = -i\{\eta^u + \eta^T \pm [(\eta^u - \eta^T)^2 + 4\eta^{uT}\eta^{Tu}]^{1/2}\}
$$

These response functions can be substituted in Eqs. (33)-(36) to produce a system of algebraic equations for corrected damping functions η. Continued iteration between Eq. (57) and Eqs. (33)-(36) produces a sequence of approximations to both the G's and the η's of increasing complexity. This procedure is loosely analogous to the representation of correlation functions by continued fractions,[16] but is evidently much less explicit.

Eq. (57) also suggests an alternate initial approximation for the response matrix to Eq. (20). Namely, substitute Eqs. (37) and (38) for η^u and η^T, and introduce the remaining damping functions

$$
\begin{aligned}
\eta^{uT}(k) &= C_D^{uT} g \\
\eta^{Tu}(k) &= C_D^{Tu} g^{-1/5} N^{2/5} k^{-6/5}
\end{aligned} \tag{58}
$$

Evaluating Eqs. (18) when $\Omega = 0$ will produce a system of equations for all of the inertial range constants analogous to the system derived from Eq. (40). The constants should be evaluated more accurately by this procedure, but closer investigation must be left to future research. From the viewpoint of the ϵ-expansion, it is easily verified that η^{uT} is $O(\epsilon)$ and that η^{Tu} is irrelevant for $\epsilon < 5$.

In outlining the higher order approximations, we have adhered to the distant interaction limit with its attendant Markovian damping. An interesting alternative is to drop this limit while retaining some kind of infrared regularization. Thus, substituting exponential response functions in Eqs. (13)-(16) makes these damping functions non-Markovian. The response functions computed from Eq. (18) with these damping functions will no longer be exponential. Similarly, the corrected forcing is not white in time. Like the iterative solution outlined above, this possibility of correcting the distant interaction limit and evaluating the correct non-Markovian damping is an interesting subject for future investigations.

V. Conclusions Bolgiano and Kolmogorov scaling have been treated as solutions of the direct interaction equations for the inertial range of buoyant turbulence. A preliminary computation of the universal inertial range constants for Bolgiano scaling has been completed. The solution entails some simplifications of the time dependence of correlation functions and response functions in the inertial range, namely that the effective forcing is white noise in time and that the effective damping is Markovian. An iterative scheme which treats the present solution as a first approximation and which corrects these hypotheses is proposed.

APPENDIX Markovianized DIA

Let the real function $H(\xi), 0 \leq \xi < \infty$ satisfy

$$H(0) = 1, \quad H(\xi) < 1 \text{ for } \xi > 0, \quad \int_0^\infty H(\xi) d\xi \ < \ \infty \tag{54}$$

Then standard properties of delta functions imply

$$\lambda H(\lambda(t-s)) \ \sim \ \delta(t-s) \text{ for } \lambda \to \infty \tag{55}$$

Rewrite Eq. (25) in the time domain, and evaluate the wavevector integrals in the distant interaction approximation in which $k \to 0, \quad p, q \to \infty$. Then

$$\begin{aligned}
\eta^u(\mathbf{k}, t, s) &= \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}}^* d\mathbf{p} d\mathbf{q} \ B(\mathbf{k}, \mathbf{p}, \mathbf{q}) G^u(p, t, s) Q^u(q, t, s) \\
&\sim \int^* d\mathbf{p} \ \{ k_m \frac{\partial B}{\partial q_m}(\mathbf{k}, \mathbf{p}, \mathbf{p}) G^u(p, t, s) Q^u(p, t, s) \\
&- B(\mathbf{k}, \mathbf{p}, \mathbf{p}) G^u(p, t, s) k_m p_m p^{-1} \frac{dQ^u}{dp}(p, t, s) \}
\end{aligned}$$

where $B(\mathbf{k}, \mathbf{p}, \mathbf{q})$ denotes the product of projection operators in Eq. (25) Assuming time stationary similarity forms $G^u(p, t, s) = G^u(p^r(t-s))$ and $Q^u(p, t, s) = R^u(p^r(t-s))Q^u(p)$, the properties Eq. (54) of H may reasonably be postulated of the product $G^u R^u$. Therefore, Eq. (54) implies that in this limit the damping is Markovian

$$\eta^u(\mathbf{k}, t, s) \;=\; \delta(t-s)\eta^u(\mathbf{k})$$

and Eq. (28) implies that the Green's function is exponential,

$$G^u(k, t, s) = \exp\,[(s-t)\eta^u(\mathbf{k})] \;\; \text{for } t \geq s \tag{55}$$

Likewise evaluating the force correlation Eq. (17) in the distant interaction limit implies that the forcing is white noise in time:

$$< f_i^u(\mathbf{k}, t) f_j^u(\mathbf{k}', s) > \;=\; \delta(t-s)\delta(\mathbf{k}+\mathbf{k}')F_{ij}^u(\mathbf{k}) \tag{56}$$

Computing the correlation function from the relation

$$\begin{aligned} Q^u(\mathbf{k}, t, s)\delta(\mathbf{k}+\mathbf{k}') \;=\; & \int_0^t dr_1 \; G^u(k, t, r_1) \int_0^t dr_2 \; G^u(k', t, r_2) \times \\ & < f^u(\mathbf{k}, r_1) f^u(\mathbf{k}', r_2) > \end{aligned}$$

using Eqs. (68), (69) shows that the fluctuation dissipation relation

$$Q^u(k, t, s) = Q^u(k)\big[G^u(k, t, s) + G^u(k, s, t)\big]$$

expressing the time dependence of the correlation functions in terms of the response function is also valid in this limit.

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